Course on "Advanced Statistical Physics"
Spring Semester (2016)
School of Nano Science, Institute for Research in Fundamental Sciences (IPM), Tehran, Iran
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Homework #9: Phase Transitions & Critical Phenomena: Gaussian fluctuations (one-loop corrections)
Due: July 15, 2016

Note: *References from which some of the following problems have been taken are available upon request.*

1: *Fluctuations around a multicritical point.* As shown in ***Homework # 8***, the Landau-Ginzburg Hamiltonian

$$\beta\mathcal{H} = \int d^d x \left[\frac{K}{2}\left(\nabla\varphi\right)^2 + \frac{t}{2}\varphi^2 + u\varphi^4 + v\varphi^6\right] \tag{1}$$

with $u = 0$ and $v > 0$ describes a *tricritical point*. Here, $\varphi(\mathbf{x})$ is a scalar field (order parameter).

(i) Compute the one-loop (or Gaussian-fluctuations) correction to the mean-field free energy.

(ii) Calculate the two-point correction function of the field fluctuations and, hence, determine the singularity in the susceptibility of the system as one approaches the critical point, $t \to 0$.

(iii) Calculate the heat capacity singularity as $t \to 0$ within the mean-field approximation and find the fluctuation correction to the heat capacity.

(iv) For $t < 0$ obtain a Ginzburg criterion and the upper critical dimension for validity of mean-field theory at a tricritical point.

(v) Use a Ginzburg criterion or dimensional analysis to find the upper critical dimension for the *multicritical point* described by the Landau-Ginzburg Hamiltonian

$$\beta\mathcal{H} = \int d^d x \left\{\frac{K}{2}(\nabla\varphi)^2 + \frac{t}{2}\varphi^2 + \frac{u_n}{n!}\varphi^n\right\}, \qquad u_n > 0. \tag{2}$$

2: Determine the upper critical dimension for a *strongly anisotropic dipolar magnet* with Hamiltonian

$$\beta\mathcal{H} = \int d^d x \left\{\frac{t}{2}\varphi^2 + u\varphi^4\right\} + \int \frac{d^d q}{(2\pi)^d}\left[cq^2 + v\left(\frac{q_\parallel^2}{q^2}\right)\right]|\varphi(\mathbf{q})|^2, \tag{3}$$

where $\parallel$ specifies the single direction defined by anisotropic field, $q^2 = q_\parallel^2 + q_\perp^2$, and u, c and v are positive constants. ***Hint:*** It is useful to introduce the anisotropic correlation lengths via:

$$G^{-1}(\mathbf{q}) = \left[t + c(q_\parallel^2 + q_\perp^2) + vq_\parallel^2/(q_\parallel^2 + q_\perp^2)\right] \simeq t\left[1 + (q_\perp\xi_\perp)^2 + v(q_\parallel\xi_\parallel/q_\perp\xi_\perp)^2\right],$$

where $\xi_\parallel = t^{-\nu_\parallel}$ and $\xi_\perp = t^{-\nu_\perp}$ with $\nu_\parallel = 1$ and $\nu_\perp = 1/2$.

3: *Gauge fluctuations in superconductors and the Anderson-Higgs mechanism.* The Ginzburg-Landau model of superconductivity describes a complex superconducting order parameter $\Psi(\mathbf{x})$ and the electromagnetic vector potential $\mathbf{A}(\mathbf{x})$ according to the Hamiltonian

$$\beta\mathcal{H} = \int d^3 x \left[\frac{t}{2}|\Psi|^2 + u|\Psi|^4 + \frac{K}{2}D_\mu\Psi D_\mu^*\Psi^* + \frac{L}{2}\left(\nabla\times\mathbf{A}\right)^2\right]. \tag{4}$$

The gauge-invariant derivative, $D_\mu = \partial_\mu - ieA_\mu(\mathbf{x})$, introduces the coupling between the two fields.

(i) Show that there is a mean-field solution of the form $\Psi(\mathbf{x}) = \overline{\Psi}$ and $\mathbf{A}(\mathbf{x}) = 0$. Find $\overline{\Psi}$ for $t > 0$ and $t < 0$.

(ii) For $t < 0$, calculate the cost of fluctuations by setting $\Psi(\mathbf{x}) = (\overline{\Psi} + \varphi(\mathbf{x}))\exp[i\theta(\mathbf{x})]$ and $A_\mu(\mathbf{x}) = a_\mu(\mathbf{x})$ (with $\partial_\mu a_\mu = 0$ in the Coulomb gauge) and expanding $\beta\mathcal{H}$ to quadratic order in φ, θ and $\mathbf{a}$.

(iii) Calculate the Fourier-transformed quantities $\langle|\varphi(\mathbf{q})|^2\rangle$, $\langle|\theta(\mathbf{q})|^2\rangle$ and $\langle|\mathbf{a}(\mathbf{q})|^2\rangle$ and show that the gauge field, which is massless in the original theory, acquires a finite mass through its coupling to the order parameter. This is known as the *Anderson-Higgs mechanism*.

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4: *Ginzburg criterion along the magnetic field direction.* Consider the Hamiltonian

$$\beta\mathcal{H} = \int d^d x \left[\frac{K}{2}(\nabla \mathbf{m})^2 + \frac{t}{2}\mathbf{m}^2 + u(\mathbf{m}^2)^2 - \mathbf{h}\cdot\mathbf{m}\right] \tag{5}$$

describing an n-component magnetization vector $\mathbf{m}(\mathbf{x})$ with $u > 0$.

(i) In the mean-field approximation indicate the resulting phase boundary in the (h, t) plane and label the phases (h denotes the magnitude of $\mathbf{h}$).

(ii) Sketch the form of the mean-field free energy density for $t < 0$, on both sides of the phase boundary, and for $t > 0$ at $h = 0$.

(iii) For t and h close to zero, the mean spontaneous magnetization can be written as $\overline{m} = t^{\beta} g_m(h/t^{\Delta})$. Identify the exponents β and Δ in the mean-field approximation.

For the remainder of this problem set $t = 0$:

(iv) Calculate the transverse and longitudinal susceptibilities at a finite h and determine the longitudinal and transverse correlation lengths.

(v) Calculate the scaling form of the one-loop correction to the free energy, magnetization and longitudinal susceptibility due to Gaussian fluctuations around the mean-field solution.

(vi) By comparing the mean-field value with the correction due to fluctuations, find the upper critical dimension, d_u, for the validity of the mean-field result.

(vii) For $d < d_u$ obtain a Ginzburg criterion by finding the field h_G below which fluctuations are important. (You may ignore the numerical coefficients in h_G, but the dependencies on K and u are required.)