Course on "Advanced Statistical Physics"
Spring Semester (2016)
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Homework #8: Phase Transitions & Critical Phenomena: Mean-field theory Due: June 13, 2016

Note: *References from which some of the following problems have been taken are available upon request.*

1: *Tricritical point.* In systems with more than one (non-ordering) control parameters, phase diagrams become multidimensional and one may encounter critical points that can be reached only by tuning two or more such control parameters. In the following example, one such point occurs in the phase diagram, where a second-order transition line meets a first-order transition line. This point is known as the tricritical point.

(i) Consider the Landau-Ginzburg "Hamiltonian"

$$\beta\mathcal{H} = \int d^d x \left[\frac{K}{2}(\nabla\varphi)^2 + \frac{t}{2}\varphi^2 + u\varphi^4 + v\varphi^6 - h\varphi \right], \tag{1}$$

where t and u can be positive or negative and $v > 0$. By sketching the (free) energy density for various t, show that in the mean-field approximation there is a first-order transition for $u < 0$ and $h = 0$ and a second-order transition for $u > 0$ and $h = 0$.

(ii) For $h = 0$ and $v > 0$, plot the phase boundaries in the (u, t) plane, identifying the phases and order of the phase transitions.

(iii) Show that $u = t = 0$ is the tricritical point in this system. For $u = 0$, calculate the tricritical exponents β, δ, γ and α (governing the singularities in magnetization, critical isotherm, susceptibility and heat capacity, respectively) and compare them with those of the scalar φ^4-model.

2: *Mean-field theory of the 3-state Potts model.* The ferromagnetic 3-state Potts model is defined by the Hamiltonian,

$$H = -J\sum_{\langle ij\rangle} \delta_{s_i s_j}, \tag{2}$$

where $J > 0$ and the indices i and j run over (nearest-neighbor) lattice sites, and the "Potts spin", s_i, takes three different values, $s_i = 1, 2, 3$.

(i) Calculate the free energy of this system within the mean-field approximation by assuming that, independently of the other spins, every spin points with probability p_q in direction $s_i = q$, where $q = 1, 2, 3$. Express the free energy density in terms of the magnetizations, $m_q = p_q - 1/3$, and show that it can be approximated by

$$\beta f \simeq \sum_{q=1}^{3} \left[\left(\frac{3}{2} - \frac{Jz}{2k_B T} \right) m_q^2 - \frac{3}{2}m_q^3 + \frac{9}{4}m_q^4 + \ldots \right], \tag{3}$$

where z is the number of nearest neighbors.

(ii) Introduce the variables, $\mu_1 = (m_2 - m_3)/2$ and $\mu_2 = (2m_1 - m_2 - m_3)/6$, and express the above free energy expression in terms of these variables. Evaluate this approximate free energy numerically as a function of these two variables and show, by producing contour plots, that the free energy develops three symmetrical minima in the ordered phase, $T < T_c$, through a *first-order transition.* Therefore, the ferromagnetic phase transition in the mean-field theory of this model is found to be of the first order!

(iii) Determine numerically the mean-field value of T_c. ***Hint:*** Consider the free energy obtained in part (i) along the special direction, $m_1 = m, m_3 = m_2 = -m/2$.

3: *Tricritical point in an antiferromagnet in a field.* An external magnetic field h in an antiferromagnet couples to the magnetization m rather than to the order parameter, which is the so-called *staggered magnetization*, m_s. Assume that the coupling between m_s and m is described phenomenologically via the free energy density

$$f = \frac{r}{2}m_s^2 + um_s^4 + \frac{r_m}{2}m^2 - hm + \frac{w}{2}m_s^2 m^2, \tag{4}$$

where $r = a(T - T^*)$, $w > 0$, and r_m is independent of temperature.

(i) Argue that the above mean-field free energy describes an antiferromagnetic transition. (***Hint:*** For $m = 0$ show that it represents a transition from $m_s = 0$ to $m_s \neq 0$; what is the transition temperature and what happens if $m_s = 0$?)

(ii) Show that this model has a tricritical point at temperature T_t and field h_t, where

$$T_t = T^* - \frac{2ur_m}{aw}, \tag{5}$$

$$h_t^2 = \frac{2ur_m^3}{w^2}. \tag{6}$$

4: *Ginzburg-Landau theory of superconductivity.* The Ginzburg-Landau theory of superconductivity addresses the interaction between a complex superconducting wave-function Ψ and a static magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$. The wave-function is not that of individual electrons, but rather that of a condensate of Cooper pairs that are pairs of loosely bound electrons. Its modulus square represents the density n_s of Cooper pairs, $n_s = |\Psi|^2$. The condensate wave-function vanishes above the critical temperature T_c and is non-zero below T_c. One postulates a thermodynamic potential

$$f - f_N = \int d^3x \left(\frac{t}{2}|\Psi|^2 + u|\Psi|^4 + \frac{1}{2m}\big| \left(-i\hbar\nabla - q\mathbf{A}\right) \Psi \big|^2 + \frac{1}{2\mu_0}B^2 - \frac{1}{\mu_0}\mathbf{H}\cdot\mathbf{B} \right), \tag{7}$$

where $\mathbf{H}$ is the induction, f_N is the thermodynamical potential of the normal phase, m and q are the mass and charge of the Cooper pairs. The coefficient b is always positive, but a changes sign at $T = T_c$ and becomes negative for $T < T_c$, so that $|\Psi|$ is non-zero below the critical temperature.

(i) Show that f is invariant under the local gauge transformation:

$$\Psi(\mathbf{x}) \to \Psi'(\mathbf{x}) = e^{-iq\Lambda(\mathbf{x})/\hbar}\Psi(\mathbf{x}), \tag{8}$$

$$\mathbf{A}(\mathbf{x}) \to \mathbf{A}'(\mathbf{x}) = \mathbf{A}(\mathbf{x}) - \nabla\Lambda(\mathbf{x}). \tag{9}$$

(ii) Show that the minimization of the free energy lead to the equations of motion,

$$\frac{1}{2m}\left(-i\hbar\nabla - q\mathbf{A}\right)^2\Psi + \frac{t}{2}\Psi + 2u\Psi|\Psi|^2 = 0, \tag{10}$$

$$-\frac{i\hbar q\mu_0}{2m}\left[\Psi^*\nabla\Psi - \Psi\nabla\Psi^*\right] - \frac{\mu_0 q^2}{m}\mathbf{A}|\Psi|^2 = \nabla\times\mathbf{B}, \tag{11}$$

and the boundary conditions,

$$\mathbf{n}\times(\mathbf{B}-\mathbf{H}) = 0, \qquad \mathbf{n}\cdot(-i\hbar\nabla - q\mathbf{A})\Psi = 0, \tag{12}$$

where $\mathbf{n}$ is a unit vector perpendicular to the surface separating the normal and the superconducting phases.

(iii) In a uniform situation, where $|\Psi|^2 = -t/(4u)$, show that superconductivity is destroyed by a magnetic induction H if

$$H^2 \geq H_c^2(T) = \frac{\mu_0 t^2}{8u}. \tag{13}$$

(iv) If $B = 0$ in a one-dimensional geometry, show that in a normal superconducting junction the order parameter increases as

$$\Psi(z) = \sqrt{\frac{|t|}{4u}}\tanh\frac{z}{\sqrt{2}\xi}, \tag{14}$$

where $\xi = (2m|t|/\hbar^2)^{-1/2}$ is the *coherence length.*