Course on "Advanced Statistical Physics"
Spring Semester (2016)
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Homework #6: Quantum Statistics Due: May 13, 2016

Note: *References from which some of the following problems have been taken are available upon request.*

1: Consider a gas of N spin-zero bosons in a d-dimensional container of volume V with a dispersion relation

$$\varepsilon_{\mathbf{p}} = \alpha |\mathbf{p}|^s, \tag{1}$$

where the constant α and the index s are both positive.

(i) Find expressions for the mean number of particles per unit volume in the ground state and the mean total number of particles in the excited states in terms of the temperature T and the fugacity $z = e^{\beta\mu}$.

(ii) Find the conditions on s and d for which Bose-Einstein condensation takes place.

(iii) Find the equation of state for this gas.

(iv) Find the relative population of the ground state N_0/N as a function of temperature, assuming that N/V is fixed.

(v) Find the entropy per unit volume of the gas in terms of T and z.

(vi) Assuming that N/V is fixed, evaluate the discontinuity in the specific heat at the critical temperature. Show that for $d = 3$ and $s = 2$, there is no discontinuity.

(vii) Evaluate the discontinuity in the derivative of the specific heat at the critical temperature for the case $d = 3$ and $s = 2$.

(viii) When there is condensation, it is possible to regard the particles in the ground state and those in excited states as two distinct phases coexisting in the same container. Find the latent heat per particle for the transition between these two phases and verify that the Clausius-Clapeyron equation is obeyed.

2: The first observation of Bose-Einstein condensation in a dilute atomic vapour was achieved by M. Anderson *et al.* [*Science* **269**, 198 (1995)]. The condensed state was produced in a vapour of ^{87}Rb that was confined by magnetic fields and was evaporatively cooled. The first evidence of condensation appeared at a temperature of about 170 nK with 2.5×10^{12} atoms cm^{-3}. Compare the condensation temperature obtained in this seminal experiment with the critical temperature for an ideal Bose gas at the same density.

3: Consider a three-dimensional gas of bosons for which the single-particle energy is given by

$$\varepsilon_{\mathbf{p},n} = \frac{|\mathbf{p}|^2}{2m} + \alpha n, \tag{2}$$

where α is a positive constant and $n = -j, \ldots, j$ is an integer.

(i) Find expressions, valid in the thermodynamic limit, for the pressure P and the mean number of particles per unit volume N/V in terms of the temperature T and the fugacity $z = e^{\beta\mu}$.

(ii) Write down the condition for Bose-Einstein condensation.

4: Consider a gas of N electrons contained in a box of volume V, whose walls have N_0 absorbent sites, each of which can absorb one electron. Let $-\varepsilon_0$ be the energy of an electron absorbed at one of these sites and $|\mathbf{p}|^2/2m$ the energy of a free electron.

(i) For $N > N_0$, find the limits as $T \to 0$ and $T \to \infty$ of the number N_a of absorbed electrons and the number N_f of free electrons.

(ii) For $N = N_0$, find the chemical potential $\mu(T)$ and the particle numbers $N_a(T)$ and $N_f(T)$ at low temperatures.

5: *Pauli Paramagnetism.* Consider a gas of N electrons contained in a box of volume V in presence of a uniform magnetic field B such that the (Zeeman) energy of an electron with spin $s_B = +1/2$ is $-\alpha B$ and the energy of an electron with $s_B = -1/2$ is αB where s_B is the component of spin in the direction of the magnetic field. Assume that the electrons are free and their kinetic energy is $|\mathbf{p}|^2/2m$.

(i) Explain what α is.

(ii) Find the number of spin $1/2$ ($N_\uparrow$) and $-1/2$ ($N_\downarrow$) at $T = 0$ in terms of the chemical potential of the gas μ.

(iii) Argue that the chemical potential of the gas at $T = 0$ is the same as its Fermi energy, ε_F.

(iv) For weak fields $|\alpha B| \ll \varepsilon_F$, show that the magnetization of the gas is

$$M = \mu_B \left(N_\uparrow - N_\downarrow\right) \approx \frac{3}{2} N \mu_B^2 \frac{B}{\varepsilon_F}. \tag{3}$$

What is μ_B?

(v) From the above, find the ground-state susceptibility of the gas.

(vi) At finite temperatures $T \neq 0$, write down an integral expression for the magnetization using the Fermi-Dirac distribution function. As you can see, this expression has an unknown, i.e., the chemical potential μ. Write down another integral expression which is supposed to determine the chemical potential.

(vii) Using Sommerfeld expansion valid at temperatures much lower than the Fermi temperature $T \ll \varepsilon_F/k_B$ approximate the integrals in the previous part. Also expand the Fermi-Dirac distribution functions for weak magnetic fields to show that the magnetization at low temperatures is

$$M(T) \approx \frac{3N\mu_B^2 B}{2\varepsilon_F}\left[1 - \frac{\pi^2}{12}\left(\frac{k_B T}{\varepsilon_F}\right)^2 + \mathcal{O}\left(\left(\frac{k_B T}{\varepsilon_F}\right)^3\right)\right]. \tag{4}$$