Course on "Statistical Physics of Fields In and Out of Equilibrium"
Spring Semester (2016)
School of Physics, Institute for Research in Fundamental Sciences (IPM), Tehran, Iran
Lecturer: Ali Naji (School of Physics, IPM)
Tutor: Bahman Roostaei (School of Physics, IPM)

Homework #9: Functional Determinants (***extra-credit problem set***)
Due: August 30, 2016

Note: *References from which the following problems have been adopted are available upon request.*

► *Functional determinants* are encountered in many physical applications of field theory and path-integral techniques, including the saddle-point evaluation of partition functions and calculation of Casimir/pseudo-Casimir forces, both topics covered in detail in this Course. In the class, we discussed a couple of different methods to evaluate functional determinants, including the *Van Vleck-Pauli-Morette* (VVPM) and *Gel'fand-Yaglom* (GY) formulas.

In Problem 1 of this homework set, the two above-mentioned methods are revisited in the context of Gaussian path integrals and are shown to be formally equivalent.

In Problem 2, we consider an alternative proof for the GY formula that was derived in the class using contour integration methods (see, e.g., K. Kirsten and A.J. McKane, "Functional determinants by contour integration methods", *Annals of Physics* **308**, 502 (2003)). The method of derivation in Problem 2 is due to Sidney Coleman (and Ian Affleck) and is outlined in a short appendix in Coleman's lectures on "The Uses of Instantons" (1977), reprinted in his *Aspects of Symmetry: Selected Erice Lectures* (Cambridge University Press, Cambridge, 1985).

1: *Equivalence of the VVPM and GY formulas.* Consider the path-integral expression for the propagator

$$K(x_f, x_i; t_f, t_i) = \mathcal{N} \int_{x(t_i)=x_i}^{x(t_f)=x_f} \mathcal{D}x(t)\, e^{iS[x(t)]/\hbar}, \tag{1}$$

with the (infinite) normalization constant and the general form of the action given respectively by

$$S[x(t)] = \int_{t_i}^{t_f} dt \left[\frac{m}{2}\dot{x}(t)^2 - V(x(t))\right], \tag{2}$$

$$\mathcal{N} = \sqrt{\frac{m}{2\pi i\hbar(t_f - t_i)}}\sqrt{\det\left(-\frac{m}{2\pi i\hbar}\,\partial_t^2\right)}. \tag{3}$$

In the semi-classical (or the saddle-point) approximation, the action is expanded up to the quadratic order in fluctuations around a classical solution $x_c(t)$ that satisfies the equation of motion $m\ddot{x}_c(t) = -V''(x_c(t))$ with the Dirichlet boundary conditions $x_c(t_i) = x_i$ and $x_c(t_f) = x_f$. Hence, the propagator is approximated by the following expression in terms of the ratio of two determinants; namely, $\det(-\partial_t^2)$, corresponding to a free-particle motion and, $\det(-\partial_t^2 - \omega(t)^2)$, corresponding to a harmonic oscillator motion with the time-dependent frequency $\omega(t)^2 = V''(x_c(t))/m$; that is,

$$K(x_f, x_i; t_f, t_i) \simeq \sqrt{\frac{m}{2\pi i\hbar(t_f - t_i)}}\sqrt{\frac{\det(-\partial_t^2)}{\det(-\partial_t^2 - \omega(t)^2)}}\, e^{iS[x_c(t)]/\hbar}. \tag{4}$$

The remarkable VVPM formula states that the ratio of the determinants in Eq. (4) can be calculated only based on the classical action as

$$\left.\frac{\det(-\partial_t^2)}{\det(-\partial_t^2 - \omega(t)^2)}\right|_{\text{VVPM}} = -\frac{t_f - t_i}{m}\cdot\frac{\partial^2 S[x_c(t)]}{\partial x_f \partial x_i}. \tag{5}$$

The GY method, on the other hand, gives the ratio of the determinants as

$$\left.\frac{\det(-\partial_t^2 - \omega(t)^2)}{\det(-\partial_t^2)}\right|_{\text{GY}} = \frac{F_\omega(t_f)}{t_f - t_i}, \tag{6}$$

where $F_\omega(t)$ satisfies the initial-value problem of a classical harmonic oscillator

$$(\partial_t^2 + \omega(t)^2)F_\omega(t) = 0, \qquad F_\omega(t_i) = 0 \quad \text{and} \quad \dot{F}_\omega(t_i) = 1. \tag{7}$$

In other words, the original *boundary-value problem* defined over the interval $[t_i, t_f]$ is reduced to an *initial-value problem* defined through a more straightforward, *homogeneous* differential equation. The remarkable point here is that, according to Eq. (6), the ratio of the determinants is given simply by the value of the solution at the ‘end point’ of the interval. The details of the GY method were discussed extensively in the class; they are revisited in Problem 2 below.

The equivalence of the VVPM and GY formulas, Eqs. (5) and (6), can be established through the following steps and by proving the *purely classical* formula

$$\frac{\partial^2 S[x_c(t)]}{\partial x_f \partial x_i} = -\frac{m}{F_\omega(t_f)}. \tag{8}$$

(i) The classical solution $x_c(t)$ with boundary conditions $x_c(t_i) = x_i$ and $x_c(t_f) = x_f$ can also be regarded as a function of the initial position and velocity, $x_c = x_c(x_i, \dot{x}_i, t)$. In the harmonic oscillator problem, one can write x_c as a superposition of two linearly independent solutions as $x_c(x_i, \dot{x}_i, t) = x_i f_1(t) + \dot{x}_i f_2(t)$. Show that

$$f_1(t_i) = 1, \dot{f}_1(t_i) = 0 \quad \text{and} \quad f_2(t_i) = 0, \dot{f}_2(t_i) = 1. \tag{9}$$

(ii) Argue that $f_2(t) = F_\omega(t)$ is the desirable GY solution.

(iii) Denote $f_1(t) = G_\omega(t)$ (as the ‘dual’ GY solution) and show that $x_c(t) = x_i G_\omega(t) + \dot{x}_i F_\omega(t)$.

(iv) Show that the classical action can be calculated in terms of the boundary values of $x_c(t)$ and $\dot{x}_c(t)$ as

$$S[x_c(t)] = \frac{m}{2}\int_{t_i}^{t_f} dt\, \left[\dot{x}_c(t)^2 - \omega(t)^2 x_c(t)^2\right] = \frac{m}{2}\left[x_f \dot{x}_c(t_f) - x_i \dot{x}_c(t_i)\right]. \tag{10}$$

(v) Using the results from part (iii) and (iv) and the fact that the Wronskian of $G_\omega(t)$ and $F_\omega(t)$ is time-independent, show that

$$S[x_c(t)] = \frac{m}{2F_\omega(t_f)}\left[\dot{F}_\omega(t_f) x_f^2 + G_\omega(t_f) x_i^2 - 2x_i x_f\right], \tag{11}$$

which immediately gives Eq. (8).

2: *Coleman’s proof of the GY formula.* The GY formula provides a powerful method to calculate the determinant of an operator $\hat{A}$ without computing its eigenvalue spectrum, $\{\lambda_n\}$. For the sake of clarity, we consider here the specific example of a one-dimensional, second-order (Schrödinger) operator $\hat{A} = -\partial_x^2 + V(x)$ defined over a finite interval $[-a, a]$ with Dirichlet boundary conditions. The eigenfunctions of $\hat{A}$, denoted by $\psi_n(x)$, satisfy the boundary-value problem

$$\left(-\partial_x^2 + V(x)\right)\psi_n(x) = \lambda_n \psi_n(x), \qquad \psi_n(-a) = \psi_n(a) = 0. \tag{12}$$

In typical physical problems, functional determinants are regularized (using, e.g., ζ-function regularization schemes as discussed in the class) or normalized with respect to a reference operator, say $\hat{A}_0 = -\partial_x^2 + V_0(x)$ (with the common choice being the ‘free-particle’ system with $V_0(x) = 0$), so as to render them finite. Therefore, we introduce the quantity

$$D(\lambda) = \frac{\det(\hat{A} - \lambda)}{\det(\hat{A}_0 - \lambda)}. \tag{13}$$

The GY formula states that the desirable (normalized) determinant $D(\lambda = 0)$ can be computed without having to solve the original boundary-value problem to establish the spectrum of the operator $\hat{A}$ (typically difficult!) and by merely finding the homogeneous solution of the initial-value problem (much simpler!)

$$\left(-\partial_x^2 + V(x)\right)\varphi_\lambda(x) = \lambda\varphi_\lambda(x), \qquad \varphi_\lambda(-a) = 0 \quad \text{and} \quad \partial_x\varphi_\lambda(-a) = 1. \tag{14}$$

The (normalized) determinant is then given by the value of the homogeneous solution $\varphi_{\lambda=0}(x)$ at $x = a$ as

$$D(\lambda = 0) = \frac{\varphi_{\lambda=0}(a)}{\varphi^{(0)}_{\lambda=0}(a)}, \tag{15}$$

which is to be proved. Note that here $\varphi^{(0)}_{\lambda=0}(x)$ is the homogeneous solution corresponding to the reference operator $\hat{A}_0$; in other words, $\varphi^{(0)}_{\lambda=0}(x)$ is obtained by setting $V \to V_0$ in Eq. (14).

(i) Show that $D(\lambda)$ is a meromorphic function with simple zeros at $\lambda = \lambda_n$ and simple poles at $\lambda = \lambda^{(0)}_n$ and that $D(\lambda) \to 1$ for $|\lambda| \to \infty$, except along the positive real axis. Here, $\{\lambda^{(0)}_n\}$ are eigenvalues of the reference operator $\hat{A}_0$, whose eigenfunctions, $\psi^{(0)}_n(x)$, satisfy a boundary-value problem analogous to Eq. (12).

(ii) Now consider the ratio

$$\Delta(\lambda) = \frac{\varphi_\lambda(a)}{\varphi^{(0)}_\lambda(a)}, \tag{16}$$

and note that $\varphi_\lambda(x)$ is an eigenfunction of $\hat{A}$ with eigenvalue λ when $\varphi_\lambda(a) = 0$. Thus, show that $\Delta(\lambda)$ is a meromorphic function with the same zeros and poles as $D(\lambda)$ and that $\Delta(\lambda) \to 1$ for $|\lambda| \to \infty$, except along the positive real axis.

(iii) Conclude that the ratio $D(\lambda)/\Delta(\lambda)$ is an analytic function that goes to 1 as $|\lambda| \to \infty$ in any direction except along the positive real axis and, therefore, it is equal to 1, leading thus to Eq. (15).