Course on "Statistical Physics of Fields In and Out of Equilibrium"
Spring Semester (2016)
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Homework #6: Continuum Spin and Quantum Systems Due: May 10, 2016

1: Consider a d-dimensional cubic lattice of a particular atom (or molecule) with each having a total spin S. The full quantum mechanical Heisenberg Hamiltonian of this system is as follows:

$$\hat{H} = J \sum_{\langle ij \rangle} \hat{\mathbf{S}}_i \cdot \hat{\mathbf{S}}_j, \tag{1}$$

in which $\hat{\mathbf{S}}_i$ is the spin vector operator at the lattice site i.

(i) Using this Hamiltonian write down the equation of motion of the spin vector at each site.

(ii) Consider the following operator transformation of a single spin known as *Holstein-Primakoff transformation*

$$\hat{S}_+ = \left(\sqrt{2S - \hat{n}}\right) \hat{b}, \tag{2}$$

$$\hat{S}_- = \hat{b}^\dagger \sqrt{2S - \hat{n}}, \tag{3}$$

$$\hat{S}_z = S - \hat{n}, \tag{4}$$

in which the operator $\hat{n} = \hat{b}^\dagger \hat{b}$. Show that $[\hat{b}, \hat{b}^\dagger] = 1$ to keep the right commutation relation for spins.

(iii) Assuming the expectation value of $\hat{n}$ is small, expand the square roots up to the first order of $\hat{n}$.

(iv) The operator $\hat{n}$ counts the number of *bosons* excited as the z-component of the spin vector evolves. Back to the cubic lattice system, consider the above transformation for each of the spins. Assume that the total quantum mechanical state of this lattice system is characterized by slow variations of the spin vector throughout the space. More specifically, assume that the z-component of each spin undergoes a slow variation throughout the space. At low energies argue that this means the time evolution of the spins is also slow. How does the slow time evolution relate to the evolution of $\hat{n}$ in time? Argue that at low energies it is possible to treat the spin operators as classical vectors.

(v) Use the result of part (i) and, by assuming that the quantum states of the system are all low energy states, show that the continuum limit of the equation of motion of each spin can be derived as follows:

$$\frac{\partial \mathbf{S}(\mathbf{x})}{\partial t} = J \mathbf{S}(\mathbf{x}) \times \nabla^2 \mathbf{S}(\mathbf{x}). \tag{5}$$

What condition(s) should J obey for such equation to be valid?

(vi) Find and sketch a wave-like solution describing small-angel precession around a globally magnetized state $\mathbf{S}(\mathbf{x}) = S\hat{\mathbf{z}}$.

(vii) Using the results of part (iii) show that for $S \gg 1$ we can always treat spins classically.

2: Consider a one-dimensional chain of atoms with equilibrium inter-atomic distance of a. The classical Lagrangian of this system is as follows:

$$L = \sum_{i=1}^{N} \left[\frac{M \dot{R}_i^2}{2} - \frac{k}{2} \left(R_{i+1} - R_i - a\right)^2 \right], \tag{6}$$

in which R_i is the position of the i-th atom in the chain and a is their equilibrium distance. M is the mass of each atom and k is the inter-atomic force constant. Assuming that the displacement from equilibrium position, denoted by u_i is small, meaning $|u_i| = |R_i - R_i^0| \ll a$, answer the following questions.

(i) Show that the Lagrangian takes the form

$$L = \sum_{i=1}^{N} \left[\frac{M \dot{u}_i^2}{2} - \frac{k}{2} \left(u_{i+1} - u_i\right)^2 \right]. \tag{7}$$

(ii) Under what condition(s) can we take the continuum limit? This limit here is taken as $u_i \to a^{1/2}u(x,t)$, $(u_{i+1}-u_i) \to a^{3/2}\partial_x u(x,t)$ and $\sum_{i=1}^{N} \to (1/a)\int dx$. Write down the Lagrangian in this limit.

(iii) Using the Lagrangian, find the momentum $\pi(x,t)$ conjugate to the displacement field $u(x,t)$ and show that the Hamiltonian of the system is as follows:

$$H[\pi,u] = \int dx \left[\frac{\pi^2}{2m} + \frac{ka^2}{2}(\partial_x u)^2\right]. \tag{8}$$

(iv) Proceed by quantizing the above system by using *quantum fields* $\hat{u}(x)$ and $\hat{\pi}(x)$ and the condition

$$[\hat{u}(x), \hat{\pi}(x')] = i\hbar\delta(x-x'). \tag{9}$$

What happened to the time dependence of the fields? Why did we assume the fields are bosonic?! Write down the Hamiltonian of this quantum field theory.

(v) Using Fourier transformation show that

$$\hat{H} = \sum_k \left[\frac{1}{2m}\hat{\pi}_k\hat{\pi}_{-k} + \frac{m\omega_k^2}{2}\hat{u}_k\hat{u}_{-k}\right], \tag{10}$$

and obtain ω_k. What are the allowed values of k? In the above, $\hat{\pi}_k$ and $\hat{u}_k$ are the Fourier transforms of the fields $\hat{\pi}(x)$ and $\hat{u}(x)$.

(vi) By defining

$$\hat{a}_k = \sqrt{\frac{m\omega_k}{2}}\left(\hat{u}_k + \frac{i}{m\omega_k}\hat{\pi}_{-k}\right), \tag{11}$$

find the commutation relation between $\hat{a}_k$ and $\hat{a}^\dagger_{k'}$ and show that the Hamiltonian reduces to an infinite number of Harmonic oscillators, the so-called *phonons*. Phonons are quasiparticles of the above Hamiltonian. What is the lifetime of these quasiparticles?

(vii) Calculate (or sketch the behavior of) the specific heat of the system as a function of temperature using bosonic statistics.