**Course on "Statistical Physics of Fields In and Out of Equilibrium"**
Spring Semester (2016)
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**Homework #4: Continuum Limit & Hubbard-Stratonovich Transformation (I)** Due: April 12, 2016

**1:** *n-vector or $O(n)$ model.* This model is defined by considering classical, $n$-component spin variables $\mathbf{S_i} = \left(S_{\mathbf{i}}^1, \ldots, S_{\mathbf{i}}^n\right)$ on the lattice sites $\mathbf{i} = (i_1, \ldots, i_d)$ of a $d$-dimensional regular lattice with the interaction Hamiltonian

$$\mathcal{H} = -\frac{1}{2}\sum_{\mathbf{i},\mathbf{j}} J_{\mathbf{ij}}\,\mathbf{S_i}\cdot\mathbf{S_j} - \sum_{\mathbf{i}} \mathbf{h_i}\cdot\mathbf{S_i}, \tag{1}$$

where $J_{\mathbf{ij}}$ is the exchange energy for given lattice sites $\mathbf{i}$ and $\mathbf{j}$ and $\mathbf{h_i}$ is the local external field. It is assumed that the spin variables at each lattice site can take continuous values with *fixed magnitude normalized* as $|\mathbf{S_i}|^2 = \sum_{\alpha=1}^{n}(S_{\mathbf{i}}^{\alpha})^2 = n$.

(i) Argue that for the cases $n = 1$, 2 and 3 the above model corresponds to the standard spin-1/2 Ising, XY and classical Heisenberg models, respectively, provided that the spin interactions are only allowed between the nearest-neighbor lattice sites.

(ii) Using the Hubbard-Stratonovich transformation show that the partition function of this model can be mapped *exactly* to the following *lattice field theory*:

$$\mathcal{Z}_N = C_J \int \left(\prod_{\mathbf{i}} d\boldsymbol{\psi}_{\mathbf{i}}\right) e^{-\beta S[\{\boldsymbol{\psi}_{\mathbf{i}}\},\{\mathbf{h_i}\}]}, \tag{2}$$

where $\boldsymbol{\psi}_{\mathbf{i}}$ is an *unconstrained* $n$-component fluctuating field at site $\mathbf{i}$ and the effective Hamiltonian ("action"):

$$S = \frac{1}{2}\sum_{\mathbf{i},\mathbf{j}} J_{\mathbf{ij}}^{-1}\,\boldsymbol{\psi}_{\mathbf{i}}\cdot\boldsymbol{\psi}_{\mathbf{j}} - \frac{1}{\beta}\ln \mathrm{tr}_{\mathbf{S}}\, e^{\beta\sum_{\mathbf{i}}(\mathbf{h_i}+\boldsymbol{\psi}_{\mathbf{i}})\cdot\mathbf{S_i}}, \tag{3}$$

where we have used the standard definition $\sum_{\mathbf{k}} J_{\mathbf{ik}}^{-1} J_{\mathbf{kj}} = \delta_{\mathbf{ij}}$ and $\mathrm{tr}_{\mathbf{S}}$ denotes the tracing (summation) over permissible spin states. Note also that $C_J \propto [\det(\beta\hat{J})]^{-1/2}$, where $\hat{J}$ is a large matrix describing the coupling between spin components across the whole lattice as required.

(iii) Derive explicit expressions for the trace-term $\mathrm{tr}_{\mathbf{S}}\, e^{\beta\sum_{\mathbf{i}}(\mathbf{h_i}+\boldsymbol{\psi}_{\mathbf{i}})\cdot\mathbf{S_i}}$ by taking a constant external field $\mathbf{h_i} = h_0\mathbf{z}$ for the cases $n = 1$, 2 and 3.

(iv) By a shift of variable $\boldsymbol{\psi}_{\mathbf{i}} \to \boldsymbol{\psi}_{\mathbf{i}} - \mathbf{h_i}$ show that:

$$\langle \mathbf{S_i}\rangle = \frac{1}{\beta}\frac{\partial \ln \mathcal{Z}_N}{\partial \mathbf{h_i}} = \sum_{\mathbf{j}} J_{\mathbf{ij}}^{-1}\langle\boldsymbol{\psi}_{\mathbf{j}}\rangle - \sum_{\mathbf{j}} J_{\mathbf{ij}}^{-1}\mathbf{h_j} \tag{4}$$

What is this result telling us about the physical meaning of the field $\boldsymbol{\psi}_{\mathbf{i}}$?

(v) For later convenience define the following field:

$$\boldsymbol{\varphi}_{\mathbf{i}} = \sum_{\mathbf{j}} J_{\mathbf{ij}}^{-1}\boldsymbol{\psi}_{\mathbf{j}} \tag{5}$$

Write down the partition function in terms of this new field (remember to keep the shifting that we performed previously). What is the physical meaning of the field $\boldsymbol{\varphi}_{\mathbf{i}}$?

(vi) In order to derive the expression in part (ii) one needs to assume that $\hat{J}$ is a positive-definite matrix. This is obviously not the case for the standard nearest-neighbor models mentioned above. Explain why?

(vii) Show that it is always possible to redefine the spin coupling matrix as $\hat{J}' = \hat{J} + J_0\hat{\mathbb{1}}$, where $J_0$ is a constant and $\hat{\mathbb{1}}$ is an identity matrix of the same dimension as $\hat{J}$, such that $\hat{J}'$ is a positive-definite matrix. You can do this by considering any of the cases $n$ =1, 2 and 3 (spin-1/2 Ising, XY and Heisenberg) explicitly. Can the foregoing procedure be used to ensure the positivity of the spin coupling matrix in the case of spin-1 Ising model?

(viii) In the continuum limit, the lattice fields $\boldsymbol{\varphi}_{\mathbf{i}}$ and $\mathbf{h}_{\mathbf{i}}$ can be mapped to continuous fields $\boldsymbol{\varphi}(\mathbf{x})$ and $\mathbf{h}(\mathbf{x})$ in space and $J_{\mathbf{ij}}$ can be mapped to a continuous interaction kernel $J(\mathbf{x}, \mathbf{x}')$. What is the relation between $\{\boldsymbol{\varphi}_{\mathbf{i}}, \mathbf{h}_{\mathbf{i}}, J_{\mathbf{ij}}\}$ and $\{\boldsymbol{\varphi}(\mathbf{x}), \mathbf{h}(\mathbf{x}), J(\mathbf{x}, \mathbf{x}')\}$?

(ix) Argue that for a homogeneous and isotropic system, $J(\mathbf{x}, \mathbf{x}') = f(|\mathbf{x} - \mathbf{x}'|)$ for a known function $f(x)$. Now assume the original interaction in the discrete model has been of the nearest-neighbor type and by expanding the function $f$ show that the field "action" can be written in general as a sum of a free-field and an interaction part, that is, in the absence of external field ($\mathbf{h} = \mathbf{0}$), $S[\boldsymbol{\varphi}] = S_0[\boldsymbol{\varphi}] + V[\boldsymbol{\varphi}]$, where $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_n)$ and

$$S_0[\boldsymbol{\varphi}] = \int d^d x \sum_{\alpha=1}^{n} \left[ \frac{K}{2} \left(\nabla \varphi_\alpha\right)^2 + \frac{t}{2} \varphi_\alpha^2 \right]. \tag{6}$$

Find the explicit form of $V[\boldsymbol{\varphi}]$ for the cases $n = 1$, 2 and 3.

(x) Show that for small $|\boldsymbol{\varphi}|$ and ($\mathbf{h} = \mathbf{0}$), the action takes the following form on the leading order:

$$S[\boldsymbol{\varphi}] = \int d^d x \sum_{\alpha=1}^{n} \left[ \frac{K}{2} \left(\nabla \varphi_\alpha\right)^2 + \frac{t}{2} \varphi_\alpha^2 + u \sum_{\beta=1}^{n} \varphi_\alpha^2 \varphi_\beta^2 \right], \tag{7}$$

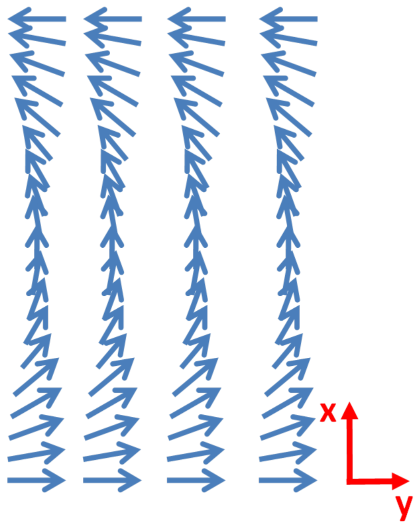
which in the Ising case ($n = 1$) reduces to the standard $\varphi^4$-theory for continuous phase transitions with a scalar order parameter. Find an expression for each of the prefactors in the above expression and show that, while $K$ and $t$ are independent of $n$, $u$ will depend on $n$.

**2:** *Field action for a helical magnetic system.* Consider a two dimensional square lattice with a classical magnetic moment $\mathbf{m}$ sitting on each site. Certain types of magnetic system show *helical order* at low temperatures because of *anisotropic next nearest neighbor interactions*

$$\mathcal{H} = -J \sum_{i,j} \mathbf{m}_{ij} \cdot \mathbf{m}_{i+1,j} - J \sum_{i,j} \mathbf{m}_{ij} \cdot \mathbf{m}_{i,j+1} + J' \sum_{i,j} \mathbf{m}_{i,j} \cdot \mathbf{m}_{i+2,j}, \tag{8}$$

in which $(i, j)$ determines a single site on the lattice and $J, J' > 0$.

(i) Why is this Hamiltonian representing an anisotropic system?



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(ii) Assuming $\mathbf{m}$ is a unit vector and the square lattice is in $x - y$ plane we can write:

$$\mathbf{m} = \{\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta\}. \tag{9}$$

Here $m_z$ represent the component of spins out of the plane of the lattice. Write down the Hamiltonian in terms of variables $\theta$ and $\varphi$. Assuming fluctuations out of the plane of the square lattice are small write down the Hamiltonian only in terms of variables $\varphi$.

(iii) With above considerations, moments are now assumed lying in the plane of the square lattice. In an anisotropic helical-order ground state the spins in one direction ($y$) are aligned, while in another direction ($x$) they make an angle $\vartheta$ with respect to each other (see Figure). Show that:

$$\vartheta = \arccos(J/4J'). \tag{10}$$

What restrictions on $J$ and $J'$ exist for the possibility of the existence of helical order?

(iv) Expand the Hamiltonian to fourth order (only in $x$-direction, *why?*) in spatial derivative to obtain the following continuum Hamiltonian:

$$\mathcal{H} \approx \frac{J}{a}\int d^2x \left\{ (\partial_y\varphi)^2 + \frac{a^2}{4}\left[(\partial_x\varphi)^2 - q^2\right]^2 + \frac{a^2}{4}(\partial_x^2\varphi)^2 \right\} \tag{11}$$

and determine $q$ in terms of $\vartheta$ and lattice constant $a$.

(v) What conditions must $\vartheta$ satisfy for the above continuum limit to be valid?