Course on "Statistical Physics of Fields In and Out of Equilibrium"
Spring Semester (2016)
School of Physics, Institute for Research in Fundamental Sciences (IPM), Tehran, Iran
Lecturer: Ali Naji (School of Physics, IPM)
Tutor: Bahman Roostaei (School of Physics, IPM)

Homework #15: Phase Transitions & Critical Phenomena: Continuous symmetry & Goldstone (zero) modes (*extra-credit problem set*)
Due: August 7, 2016

Note: *References from which some of the following problems have been taken are available upon request.*

1: *Spin waves.* Consider a classical spin model defined on a regular lattice by a two-component spin *unit* vector $\mathbf{S} = (S_x, S_y)$ and the nearest-neighbor interaction Hamiltonian

$$\beta\mathcal{H} = -K\sum_{\langle ij\rangle} \mathbf{S}_i \cdot \mathbf{S}_j. \tag{1}$$

(i) Write down the partition function of the system as an integral over the set of angles $\{\theta_i\}$ between the spins $\{\mathbf{S}_i\}$ and some arbitrary axis.

(ii) At low temperatures ($K \gg 1$), these angles vary slowly from site to site. In this case, expand the Hamiltonian to get a quadratic form in $\{\theta_i\}$.

(iii) In one dimension, consider N sites with periodic boundary conditions and find the normal (Fourier) modes, $\{\theta_q\}$, that diagonalize the quadratic form and their corresponding eigenvalues.

(iv) Generalize the results from part (iii) to a d-dimensional simple cubic lattice with periodic boundary conditions.

(v) Calculate the contribution of these modes to the free energy and heat capacity.

(vi) Find an expression for $\langle \mathbf{S}_0 \cdot \mathbf{S}_\mathbf{x}\rangle = \langle\cos(\theta_\mathbf{x} - \theta_0)\rangle$ by adding contributions from different Fourier modes. Convince yourself that, for $|\mathbf{x}| \to \infty$, only $|\mathbf{q}| \to 0$ modes contribute appreciably to this expression and, hence, calculate the asymptotic limit.

(vii) Calculate the transverse susceptibility from $\chi_t \propto \int d^d\mathbf{x}\, \langle \mathbf{S}_0 \cdot \mathbf{S}_\mathbf{x}\rangle$; how does it depend on the system size N?

(viii) In two dimensions, show that χ_t only diverges for K larger than a critical value $K_c = 1/(4\pi)$.

2: *One-loop corrections in the presence of Goldstone (zero) modes.* Consider the partition function

$$\mathcal{Z} = \int \mathcal{D}\varphi\, e^{-S[\varphi(x)]} \tag{2}$$

in the presence of a continues symmetry for which the mean-field equation

$$\left.\frac{\delta S[\varphi(x)]}{\delta\varphi(x)}\right|_{\varphi=\varphi_c(x;\boldsymbol{\theta})} = 0 \tag{3}$$

has degenerate solutions $\varphi_c(x;\boldsymbol{\theta})$ parametrized by an n-component vector $\boldsymbol{\theta} = \{\theta_1, \dots, \theta_n\}$. Expansion of the above integral around a specific mean field, $\varphi_c(x,\theta)$, yields

$$\mathcal{Z} \simeq e^{-S[\varphi_c(x;\boldsymbol{\theta})]} \int \mathcal{D}\varphi\, e^{-\frac{1}{2}\int dxdy\, [\varphi(x)-\varphi_c(x;\boldsymbol{\theta})]A(x,y)[\varphi(y)-\varphi_c(y;\boldsymbol{\theta})]}, \tag{4}$$

where

$$A(x,y) = \left.\frac{\delta^2 S}{\delta\varphi(x)\delta\varphi(y)}\right|_{\varphi=\varphi_c(x;\boldsymbol{\theta})}. \tag{5}$$

(i) By taking derivatives of Eq. (3) w.r.t. θ_i, show that there are n Goldstone modes $\partial\varphi_c(x;\boldsymbol{\theta})/\partial\theta_i$ satisfying

$$\int dy\, A(x,y)\frac{\partial\varphi_c(y;\boldsymbol{\theta})}{\partial\theta_i} = 0. \tag{6}$$

(ii) Consider the functions $f_i(\boldsymbol{\theta})$ defined as

$$f_i(\boldsymbol{\theta}) \equiv \int dx\, \frac{\partial\varphi_c(x;\boldsymbol{\theta})}{\partial\theta_i}\left[\varphi_c(x;\boldsymbol{\theta}) - \varphi(x)\right]. \tag{7}$$

Then, the fluctuations orthogonal to the Goldstone modes satisfy $f_i(\boldsymbol{\theta}) = 0$ for $i = 1, \ldots, n$. Show that these constraints may be included in the functional integral by including the following identity in the partition function:

$$\int \prod_{i=1}^{n} d\theta_i \prod_{i=1}^{n} \delta\left(f_i(\boldsymbol{\theta})\right) \det\left(\frac{\partial f_i}{\partial \theta_j}\right) = 1. \tag{8}$$

(iii) Introducing the auxiliary field $h(x)$, show that the partition function in Eq. (4) is given by

$$\mathcal{Z} \simeq e^{-S[\varphi_c(x;\boldsymbol{\theta})]} \det\left(\int dx\, \frac{\partial \varphi_c}{\partial \theta_i}\frac{\partial \varphi_c}{\partial \theta_j} - \int dx\, \frac{\partial^2 \varphi_c}{\partial \theta_i \partial \theta_j}\frac{\delta}{\delta h(x)}\right) \mathcal{L}[h]\bigg|_{h=0}, \tag{9}$$

where we have defined

$$\mathcal{L}[h] \equiv \int \prod_{i=1}^{n} \frac{d\lambda_i d\theta_i}{2\pi} \int \mathcal{D}\varphi\, e^{-\frac{1}{2}\int dx dy\, \varphi(x) A(x,y) \varphi(y) + \int dx\, \varphi(x)\left(-i\sum_{i=1}^{n} \lambda_i \frac{\partial \varphi_c}{\partial \theta_i} + h(x)\right)}. \tag{10}$$

(iv) Since the Goldstone modes, $\{|\frac{\partial \varphi_c}{\partial \theta_i}\rangle\}$, are not necessarily orthonormal, let $\{|\psi_i\rangle\}$ for $i = 1, \ldots n$ denote an orthonormal basis of the zero eigenspace of A and define

$$A_\epsilon \equiv \epsilon \sum_{i=1}^{n} |\psi_i\rangle\langle\psi_i| + A_\perp, \tag{11}$$

where ϵ is a positive infinitesimal number, which is eventually taken to zero, and $A_\perp$ is the projection of A on the subspace orthogonal to the zero modes. Replacing A by A_ϵ in Eq. (10) and performing the φ integral, show that when ϵ goes to zero, we have

$$\mathcal{L} \simeq \frac{\epsilon^{-n/2}}{\sqrt{\det A_\perp}} \int \prod_{i=1}^{n} \frac{d\lambda_i d\theta_i}{2\pi}\, e^{\frac{1}{2\epsilon}\sum_{k=1}^{n}\left(\langle h|\psi_k\rangle - i\sum_{i=1}^{n} \lambda_i \left\langle \frac{\partial \varphi_c}{\partial \theta_i}|\psi_k\right\rangle\right)^2}, \tag{12}$$

where $\langle f|g\rangle$ denotes $\int dx\, f(x) g(x)$.

(v) Performing the $\{\lambda_i\}$ integrals in Eq. (12), show that

$$\mathcal{L} \simeq \frac{1}{\sqrt{\det A_\perp}} \int \prod_{i=1}^{n} \frac{d\theta_i}{\sqrt{2\pi}} \left(\det\left\langle \frac{\partial \varphi_c}{\partial \theta_i}\middle|\frac{\partial \varphi_c}{\partial \theta_j}\right\rangle\right)^{-1/2}, \tag{13}$$

and thus

$$\mathcal{Z} \simeq \frac{e^{-S[\varphi_c(x;\boldsymbol{\theta})]}}{\sqrt{\det A_\perp}} \int \prod_{i=1}^{n} \frac{d\theta_i}{\sqrt{2\pi}} \left(\det\left\langle \frac{\partial \varphi_c}{\partial \theta_i}\middle|\frac{\partial \varphi_c}{\partial \theta_j}\right\rangle\right)^{1/2}. \tag{14}$$

(vi) Generalize the above argument in the case of a vector field, $\boldsymbol{\varphi}(x)$.

3: *Perturbation theory in the presence of Goldstone (zero) modes.* In order to perform a perturbative expansion, it is sufficient to be able to calculate Gaussian integrals using the standard identity

$$\begin{aligned}\mathcal{Z} &= \int \mathcal{D}\varphi\, e^{-\frac{1}{2}\int dx dy\, \varphi(x) A(x,y) \varphi(y) - \int dx\, V(\varphi(x))} \\ &= e^{-\int dx\, V\left(\frac{\delta}{\delta h(x)}\right)} \int \mathcal{D}\varphi\, e^{-\frac{1}{2}\int dx dy\, \varphi(x) A(x,y) \varphi(y) + \int dx\, h(x) \varphi(x)}\bigg|_{h=0}.\end{aligned} \tag{15}$$

Spring Semester (2016)
School of Physics, Institute for Research in Fundamental Sciences (IPM), Tehran, Iran
Lecturer: Ali Naji (School of Physics, IPM)
Tutor: Bahman Roostaei (School of Physics, IPM)

Homework #15: Phase Transitions & Critical Phenomena: Continuous symmetry & Goldstone (zero) modes (*extra-credit problem set*) Due: August 7, 2016

Let us define

$$\mathcal{L}[h] \equiv \int \mathcal{D}\varphi \, e^{-\frac{1}{2}\int dxdy\,[\varphi(x)-\varphi_c(x;\theta_0)]A(x,y)[\varphi(y)-\varphi_c(y;\theta_0)]+\int dx\,h(x)\varphi(x)}, \tag{16}$$

and $\partial\varphi_c \equiv \frac{\partial\varphi_c}{\partial\theta_0}$ as a zero eigenvalue of $A(x,y)$. We regularize A as

$$A = \epsilon\frac{|\partial\varphi_c\rangle\langle\partial\varphi_c|}{\langle\partial\varphi_c|\partial\varphi_c\rangle} + A_\perp, \tag{17}$$

and integrate over transverse fluctuations by introducing in Eq. (16) the identity

$$\int d\theta_0\, f'(\theta_0)\delta\left[f(\theta_0)\right] = 1, \tag{18}$$

where

$$f(\theta_0) \equiv \int dx\, \frac{\partial\varphi_c(x;\theta_0)}{\partial\theta_0}\left[\varphi_c(x;\theta_0) - \varphi(x)\right]. \tag{19}$$

(i) Show that

$$\mathcal{L}[h] = \int dx\, \left[\left(\frac{\partial\varphi_c(x)}{\partial\theta_0}\right)^2 - \frac{\partial^2\varphi_c}{\partial\theta_0^2}\frac{\delta}{\delta h(x)}\right]\mathcal{L}_1[h], \tag{20}$$

where we have defined

$$\mathcal{L}_1[h] \equiv \frac{1}{\sqrt{\det A_\perp}}\int \frac{d\theta_0}{\sqrt{2\pi}}\frac{e^{\frac{1}{2}\int dxdy\, h_\perp(x)A_\perp^{-1}(x,y)h_\perp(y)}}{\sqrt{\int dx\,(\frac{\partial\varphi_c}{\partial\theta_0})^2}}, \tag{21}$$

and

$$h_\perp(x) \equiv h(x) - \frac{\langle\partial\varphi_c|h\rangle}{\langle\partial\varphi_c|\partial\varphi_c\rangle}\frac{\partial\varphi_c(x)}{\partial\theta_0}. \tag{22}$$

(ii) Show that

$$\mathcal{L}[h] = \left[\int dx\,\left(\frac{\partial\varphi_c(x;\theta_0)}{\partial\theta_0}\right)^2 - \int dxdy\,\frac{\partial^2\varphi_c(x;\theta_0)}{\partial\theta_0^2}A_\perp^{-1}(x,y)h_\perp(y)\right]\mathcal{L}_1[h], \tag{23}$$

where we have used the fact that

$$A_\perp^{-1}|\partial\varphi_c\rangle = 0. \tag{24}$$

(iii) By expressing $\frac{\delta}{\delta h(x)}$ in terms of $\frac{\delta}{\delta h_\perp(x)}$, write the formal perturbative expansion of $\mathcal{Z}$ as the exponential of an operator $\left(\frac{\delta}{\delta h_\perp(x)}\right)$ acting on $\mathcal{L}[h]$ at $h = 0$. Evaluate the first-order term of this perturbative expansion.