**Course on "Statistical Physics of Fields In and Out of Equilibrium"**
Spring Semester (2016)
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**Homework #14: Phase Transitions & Critical Phenomena: Momentum-space RG** Due: July 30, 2016

Note: *References from which the following problem has been adopted are available upon request.*

**1:** Consider the Landau-Ginzburg Hamiltonian involving a two-component order parameter $\boldsymbol{\varphi} = (\varphi_1, \varphi_2)$:

$$\beta\mathcal{H}[\boldsymbol{\varphi}] = \sum_{i=1}^{2} \int d^d x \left\{ \frac{1}{2}(\nabla\varphi_i)^2 + \frac{r}{2}\varphi_i^2 + g_1\varphi_i^4 \right\} + g_2 \int d^d x\, \varphi_1^2\varphi_2^2. \tag{1}$$

This model has the appropriate symmetry to describe a structural phase transition in a crystal with a fourth-order axis. The order parameters $\varphi_1$ and $\varphi_2$ may be interpreted as a projection of the atomic position onto a coordinate system normal to this axis.

(i) Assuming $g_1 > 0$ and putting $\Gamma = g_2/g_1$, study the different phases of the system as a function of $r$ and $\Gamma$ within the mean-field (saddle-point) approximation. Show that for $r < 0$, spontaneous symmetry breaking occurs, such that

$$\begin{aligned} \Gamma > 2: &\ \ \varphi_1 \neq 0, \varphi_2 = 0 \ \text{ or } \ \varphi_2 \neq 0, \varphi_1 = 0, \\ -2 < \Gamma < 2: &\ \ \varphi_1 = \pm\varphi_2. \end{aligned} \tag{2}$$

Show that the system is thermodynamically unstable for $\Gamma < -2$. Draw a phase diagram in the $r - \Gamma$ plane, indicating phase boundaries of first- and second-order phase transitions.

(ii) Using Wilson's perturbative momentum-space RG analysis (in three standard steps of *coarse-graining*, *rescaling* and *renormalizing*), and taking advantage of Feynman's diagrammatic representation, derive the following RG equations for the coupling constants $g_1$ and $g_2$:

$$\begin{aligned} \frac{dg_1}{d\tau} &= \epsilon g_1 - A(36g_1^2 + g_2^2), \\ \frac{dg_2}{d\tau} &= \epsilon g_2 - A(24g_1g_2 + 8g_2^2), \end{aligned} \tag{3}$$

where $A$ is a constant, $\epsilon = 4 - d$ and we have defined $\tau = \log \ell$, where $\ell$ is the factor by which the unit of length is increased within the RG transformation.

(iii) Using the dimensionless parameters

$$\rho = r\Lambda^{-2}, \ \Gamma = g_2/g_1, \ \gamma = g_1\Lambda^{-\epsilon}, \tag{4}$$

where $\Lambda$ is a UV cut-off (and after an appropriate redefinition of the parameters), show that

$$\begin{aligned} \frac{d\rho}{d\tau} &= 2\rho + 12(1-\rho)(1+\Gamma/6)\gamma, \\ \frac{d\Gamma}{d\tau} &= \Gamma(\Gamma-2)(\Gamma-6)\gamma, \\ \frac{d\gamma}{d\tau} &= \gamma[\epsilon - \gamma(36+\Gamma^2)]. \end{aligned} \tag{5}$$

(iv) Using the method of $\epsilon$-expansion, investigate the fixed points and their stabilities and determine the corresponding critical exponents to first order in $\epsilon$.

(v) Discuss possible generalization of the above problem to the case of an $n$-component order parameter $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_n)$. Note that this generalization leads to an *anisotropic n-vector model* with the Landau-Ginzburg Hamiltonian

$$\beta\mathcal{H}[\boldsymbol{\varphi}] = \int d^d x \left\{ \sum_i \left[ \frac{1}{2}(\nabla\varphi_i)^2 + \frac{r}{2}\varphi_i^2 \right] + u\big(\sum_i \varphi_i^2\big)^2 + v\sum_i \varphi_i^4 \right\}. \tag{6}$$

**For extra credit:** Apply the RG transformation to the above Hamiltonian, investigate the fixed points and their stabilities, and determine the corresponding critical exponents for the anisotropic $n$-vector model.