

Spring Semester (2016)

School of Physics, Institute for Research in Fundamental Sciences (IPM), Tehran, Iran

Lecturer: Ali Naji (School of Physics, IPM)

Tutor: Bahman Roostaei (School of Physics, IPM)

Homework #12: Phase Transitions & Critical Phenomena: Gaussian fluctuations (one-loop corrections)

Due: July 15, 2016

Note: *References from which some of the following problems have been taken are available upon request.*

1: Fluctuations around a multicritical point. As shown in **Homework # 11**, the Landau-Ginzburg Hamiltonian

$$\beta\mathcal{H} = \int d^d x \left[\frac{K}{2} (\nabla\varphi)^2 + \frac{t}{2}\varphi^2 + u\varphi^4 + v\varphi^6 \right] \quad (1)$$

with $u = 0$ and $v > 0$ describes a *tricritical point*. Here, $\varphi(\mathbf{x})$ is a scalar field (order parameter).

- (i) Compute the one-loop (or Gaussian-fluctuations) correction to the mean-field free energy.
- (ii) Calculate the two-point correction function of the field fluctuations and, hence, determine the singularity in the susceptibility of the system as one approaches the critical point, $t \rightarrow 0$.
- (iii) Calculate the heat capacity singularity as $t \rightarrow 0$ within the mean-field approximation and find the fluctuation correction to the heat capacity.
- (iv) For $t < 0$ obtain a Ginzburg criterion and the upper critical dimension for validity of mean-field theory at a tricritical point.
- (v) Use a Ginzburg criterion or dimensional analysis to find the upper critical dimension for the *multicritical point* described by the Landau-Ginzburg Hamiltonian

$$\beta\mathcal{H} = \int d^d x \left\{ \frac{K}{2} (\nabla\varphi)^2 + \frac{t}{2}\varphi^2 + \frac{u_n}{n!}\varphi^n \right\}, \quad u_n > 0. \quad (2)$$

2: Determine the upper critical dimension for a *strongly anisotropic dipolar magnet* with Hamiltonian

$$\beta\mathcal{H} = \int d^d x \left\{ \frac{t}{2}\varphi^2 + u\varphi^4 \right\} + \int \frac{d^d q}{(2\pi)^d} \left[cq^2 + v \left(\frac{q_{\parallel}^2}{q^2} \right) \right] |\varphi(\mathbf{q})|^2, \quad (3)$$

where \parallel specifies the single direction defined by anisotropic field, $q^2 = q_{\parallel}^2 + q_{\perp}^2$, and u , c and v are positive constants. **Hint:** It is useful to introduce the anisotropic correlation lengths via:

$$G^{-1}(\mathbf{q}) = \left[t + c(q_{\parallel}^2 + q_{\perp}^2) + vq_{\parallel}^2/(q_{\parallel}^2 + q_{\perp}^2) \right] \simeq t \left[1 + (q_{\perp}\xi_{\perp})^2 + v(q_{\parallel}\xi_{\parallel}/q_{\perp}\xi_{\perp})^2 \right],$$

where $\xi_{\parallel} = t^{-\nu_{\parallel}}$ and $\xi_{\perp} = t^{-\nu_{\perp}}$ with $\nu_{\parallel} = 1$ and $\nu_{\perp} = 1/2$.

3: Gauge fluctuations in superconductors and the Anderson-Higgs mechanism. The Ginzburg-Landau model of superconductivity describes a complex superconducting order parameter $\Psi(\mathbf{x})$ and the electromagnetic vector potential $\mathbf{A}(\mathbf{x})$ according to the Hamiltonian

$$\beta\mathcal{H} = \int d^3 x \left[\frac{t}{2}|\Psi|^2 + u|\Psi|^4 + \frac{K}{2}D_{\mu}\Psi D_{\mu}^*\Psi^* + \frac{L}{2}(\nabla \times \mathbf{A})^2 \right]. \quad (4)$$

The gauge-invariant derivative, $D_{\mu} = \partial_{\mu} - ieA_{\mu}(\mathbf{x})$, introduces the coupling between the two fields.

- (i) Show that there is a mean-field solution of the form $\Psi(\mathbf{x}) = \bar{\Psi}$ and $\mathbf{A}(\mathbf{x}) = 0$. Find $\bar{\Psi}$ for $t > 0$ and $t < 0$.
- (ii) For $t < 0$, calculate the cost of fluctuations by setting $\Psi(\mathbf{x}) = (\bar{\Psi} + \varphi(\mathbf{x})) \exp[i\theta(\mathbf{x})]$ and $A_{\mu}(\mathbf{x}) = a_{\mu}(\mathbf{x})$ (with $\partial_{\mu}a_{\mu} = 0$ in the Coulomb gauge) and expanding $\beta\mathcal{H}$ to quadratic order in φ , θ and \mathbf{a} .
- (iii) Calculate the Fourier-transformed quantities $\langle |\varphi(\mathbf{q})|^2 \rangle$, $\langle |\theta(\mathbf{q})|^2 \rangle$ and $\langle |\mathbf{a}(\mathbf{q})|^2 \rangle$ and show that the gauge field, which is massless in the original theory, acquires a finite mass through its coupling to the order parameter. This is known as the *Anderson-Higgs mechanism*.

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4: *Ginzburg criterion along the magnetic field direction.* Consider the Hamiltonian

$$\beta\mathcal{H} = \int d^d x \left[\frac{K}{2} (\nabla \mathbf{m})^2 + \frac{t}{2} \mathbf{m}^2 + u (\mathbf{m}^2)^2 - \mathbf{h} \cdot \mathbf{m} \right] \quad (5)$$

describing an n -component magnetization vector $\mathbf{m}(\mathbf{x})$ with $u > 0$.

(i) In the mean-field approximation indicate the resulting phase boundary in the (h, t) plane and label the phases (h denotes the magnitude of \mathbf{h}).

(ii) Sketch the form of the mean-field free energy density for $t < 0$, on both sides of the phase boundary, and for $t > 0$ at $h = 0$.

(iii) For t and h close to zero, the mean spontaneous magnetization can be written as $\overline{m} = t^\beta g_m(h/t^\Delta)$. Identify the exponents β and Δ in the mean-field approximation.

For the remainder of this problem set $t = 0$:

(iv) Calculate the transverse and longitudinal susceptibilities at a finite h and determine the longitudinal and transverse correlation lengths.

(v) Calculate the scaling form of the one-loop correction to the free energy, magnetization and longitudinal susceptibility due to Gaussian fluctuations around the mean-field solution.

(vi) By comparing the mean-field value with the correction due to fluctuations, find the upper critical dimension, d_u , for the validity of the mean-field result.

(vii) For $d < d_u$ obtain a Ginzburg criterion by finding the field h_G below which fluctuations are important. (You may ignore the numerical coefficients in h_G , but the dependencies on K and u are required.)