Course on "Statistical Physics of Fields In and Out of Equilibrium"
Spring Semester (2016)
School of Physics, Institute for Research in Fundamental Sciences (IPM), Tehran, Iran
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Homework #10: Casimir Effect & Fluctuation-Induced Interactions Due: May 17, 2016

1: *Zero-frequency Lifshitz-Casimir-Van der Waals interactions.* The zero-frequency (or thermal) Lifshitz-Casimir-Van der Waals interactions between neutral dielectric objects are generated by the zero-frequency Matsubara modes of the electromagnetic field fluctuations. The latter, described by a scalar field, are dominant in the so-called classical (or high-temperature) regime, where the distance between juxtaposed dielectric surfaces is comparable to or larger than the thermal wavelength (which is around 8 μm at room temperature).[1]

Let us consider two semi-infinite slabs of dielectric constants ε_1 and ε_2 with plane-parallel inner surfaces (of infinite area) at separation distance d. The slabs are immersed in a continuum medium of dielectric constant ε_m, giving thus an inhomogeneous spatial dielectric profile as

$$\varepsilon(\mathbf{r}) = \begin{cases} \varepsilon_1 & z < -d/2, \\ \varepsilon_m & |z| < d/2, \\ \varepsilon_2 & z > d/2. \end{cases} \tag{1}$$

(i) Write down the zero-frequency field action and the corresponding partition function for this system. Justify your answer by considering the full electromagnetic field action in the zero-frequency limit.

(ii) Calculate the free energy of the system (using any one of the calculation methods discussed in the class) and apply an appropriate regularization scheme to show that the Lifshitz-Casimir-Van der Waals interaction pressure acting on each of the slabs is given by

$$P(d) = -k_B T \times \frac{\mathrm{Li}_3(\Delta_1 \Delta_2)}{8\pi d^3} \qquad \text{where} \quad \Delta_i = \frac{\varepsilon_i - \varepsilon_m}{\varepsilon_i + \varepsilon_m}, \tag{2}$$

is the dielectric discontinuity parameter at the bounding surfaces (labelled by $i = 1, 2$) and $\mathrm{Li}_3(\cdot)$ is the trilogarithm function.

(iii) Discuss the situations in which the interaction pressure can be attractive or repulsive.

(iv) Assume that the intervening region between the slabs contains a classical plasma, such as a symmetric Coulomb fluid consisting of two positive and negative ionic species with charges $\pm e_0$ in equilibrium with a bulk reservoir of ionic concentration n_b; the latter lead to Debye screening at length scales beyond κ^{-1} (where we have standardly defined $\kappa^2 = 8\pi \ell_B n_b$ and $\ell_B = e_0^2/(4\pi\varepsilon_m\varepsilon_0 k_B T)$) and, hence, a *massive scalar field theory.* Generalize the results in parts (i)-(iii) to account for the screening ("mass") effects.

(v) Assume that, in the original problem discussed in parts (i)-(iii), the two plane-parallel dielectric slabs have finite thicknesses of δ_1 and δ_2. Based on general considerations, argue that the Lifshitz-Casimir-Van der Waals interaction pressure should scale with the inter-surface separation (and slab thicknesses) as

$$P(d) \sim -\frac{k_B T}{d^3} \times \begin{cases} 1 & d \ll \delta_1, \delta_2, \\ \delta_1/d & \delta_1 \ll d \ll \delta_2, \\ \delta_1\delta_2/d^2 & \delta_1, \delta_2 \ll d. \end{cases} \tag{3}$$

The interaction pressure thus becomes of shorter range (and strength) when the finiteness of the slab thicknesses becomes important (*Why?*).

(vi) Calculate the interaction free energy and pressure as a function of the inter-surface distance, d, when the slabs have a finite thickness and systematically derive the limiting results shown in part (v).

[1] Note, however, that the zero-frequency mode starts to give a sizable contribution as compared to the sum of non-zero Matsubara frequencies already at much smaller separations and, when the dielectric bodies are immersed in certain media, such as water which has peculiar dielectric properties, the zero-frequency contribution can become large and even dominant at separation distances of tens to hundreds of nanometers.

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2: *Pseudo-Casimir effect in nematic liquids.* Liquid crystals are among the most extensively studied examples of anisotropic correlated fluids. This is partly due to the fact that they are easily accessible for experimental investigation at room temperature and require no fine-tuning in order to achieve criticality. In the nematic phase, characterized by spontaneous breaking of two continuous rotational symmetries, liquid crystals exhibit orientational (but *no positional*) ordering that can be described in terms of a director (*order parameter*) field, $\mathbf{n}(\mathbf{r})$, giving the local preferred direction of the long axis of the molecules.

Consider a nematic film bounded by two rigid, plane-parallel surfaces located at separation distance d normal to the z-axis (see Fig. 1). The surfaces are assumed to impose a *strong homeotropic anchoring*, meaning that the director field is constrained to be in the normal direction at the bounding surfaces. The nematic phase can thus be characterized by a uniform *mean* director field $\mathbf{n}_0 = \mathbf{z}$. The cost of thermal fluctuations, $\delta\mathbf{n}$, of the nematic director around this mean orientation, i.e., for $\mathbf{n} = \mathbf{n}_0 + \delta\mathbf{n}$, is given by the Frank's continuum elastic energy

$$H = \frac{1}{2}\int d\mathbf{r}\,[K_1(\nabla\cdot\mathbf{n})^2 + K_2(\mathbf{n}\cdot\nabla\times\mathbf{n})^2 + K_3(\mathbf{n}\times\nabla\times\mathbf{n})^2], \tag{4}$$

where K_1, K_2, and K_3 are the splay, twist, and bend elastic constants, respectively.

(i) Show that, up to the second order in field fluctuations, the above elastic energy expression can be written in terms of two scalar fluctuation fields, $\{\varphi_1(\mathbf{r}), \varphi_2(\mathbf{r})\}$, corresponding to the two massless (Goldstone) modes that result from spontaneous breaking of the continuous rotational symmetries as

$$H = \frac{1}{2}\sum_{i=1,2}\int d\mathbf{r}\,[K_3(\partial_z\varphi_i)^2 + K_i(\nabla_\perp\varphi_i)^2], \tag{5}$$

where $\mathbf{r} = (\boldsymbol{\rho}, z)$ and $\nabla_\perp \equiv \partial/\partial\boldsymbol{\rho}$ for the transverse directions $\boldsymbol{\rho} = (x, y)$.

(ii) Show that the nematic fluctuations lead to a pseudo-Casimir (or Casimir-like) interaction pressure, which is attractive and is given by the expression

$$P(d) = -k_BT\frac{\zeta(3)}{8\pi d^3}\left(\frac{K_3}{K_1} + \frac{K_3}{K_2}\right). \tag{6}$$

(iii) Discuss how *repulsive* fluctuation-induced interactions may arise in this system. **Hint:** Consider surfaces with *unlike* boundary conditions!

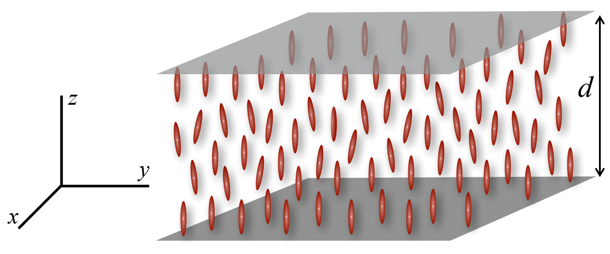

Figure 1. Schematic view of a nematic liquid-crystalline film confined between two rigid, plane-parallel surfaces at separation distance d with normal boundary conditions for the director field at both surfaces. Thermal fluctuations lead to small deviations from the uniform mean director field along the z-axis and lead to a long-range, attractive pseudo-Casimir interaction between the bounding surfaces.